Isabel Restrepo
Brown University
January 11, 2011

1 Tylor expansion of Images

The promblem of computing mage derivatives arises from a least square error minimization of the energy function (1-d case)

$$\mathrm{E} = \sum_{i=-n}^{n} (I(n) - \tilde{I}(n))^2 \tag{1}$$

Where $\tilde{I}$ is the Taylor series approximation of I.
Recall that the Taylor series expansion of a function $f(x)$ about a point a is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n$$

1.1 The 2-d case

Consider the second order Tylor series expansion about a point (x_0, y_0) in a 2-d image

$$\begin{aligned} \tilde{I}(x,y) &= I + I_x \cdot (x - x_0) + I_y \cdot (y - y_0) && (2) \\ &+ \frac{1}{2}[I_{xx} \cdot (x-x_0)^2 + 2 \cdot I_{xy} \cdot (x-x_0)(y-y_0) + I_{yy} \cdot (y-y_0)^2] && (3) \end{aligned}$$

Where $I_x = I_x(x_0, y_0)$ has been used to keep the notation short.

This expansion can be expressed in vector-matrix form as,

$$\tilde{I}(x,y) = \begin{bmatrix} (x-x_0) & (y-y_0) & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} I_{xx} & I_{xy} & I_x \\ I_{xy} & I_{yy} & I_y \\ I_x & I_y & 2I \end{bmatrix} \begin{bmatrix} (x-x_0) \\ (y-y_0) \\ 1 \end{bmatrix}$$

1.2 The 3-d case

In general, the second-order Taylor series expansion, about a d-dimensional point $\mathbf{a}$, of multivariate functions can be written as,

$$\tilde{f}(\mathbf{x}) = f(\mathbf{a}) + (\mathbf{x}-\mathbf{a})^{\mathrm{T}} \nabla f(\mathbf{a}) + \frac{1}{2!}(\mathbf{x}-\mathbf{a})^{\mathrm{T}} \{\nabla\nabla f(\mathbf{a})\} (\mathbf{x}-\mathbf{a})$$

Where, $\nabla f(\mathbf{a})$ is the gradiant of f evaluated at $\mathbf{x} = \mathbf{a}$, and $\nabla\nabla f(\mathbf{a})$ is the Hessian matrix of f evaluated at $\mathbf{x} = \mathbf{a}$,
So, for a 3-d image we obtain

$$\tilde{I}(\mathbf{x}) = I(\mathbf{a}) + (\mathbf{x}-\mathbf{a})^{\mathrm{T}} \begin{bmatrix} I_x(\mathbf{a}) \\ I_y(\mathbf{a}) \\ I_z(\mathbf{a}) \end{bmatrix} + \frac{1}{2!}(\mathbf{x}-\mathbf{a})^{\mathrm{T}} \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} (\mathbf{x}-\mathbf{a})$$

1.3 Kernel Approximation

Ideally the expansion would agree exactly with intensity values, but since this is not the case, the best we can do can be found with a least squares approximation.

$$E = \sum_{i=-ni}^{ni} \sum_{j=-nj}^{nj} \sum_{k=-nk}^{nk} \left(I(i,j,k) - \tilde{I}(i,j,k) \right)^2$$

1.3.1 Notation and assumptions

Assume $ni = nj = nk$
Assume $\mathbf{a} = \mathbf{0}$
$I(\mathbf{x}) = I(i,j,k)$
$\mathrm{G} = \nabla f(\mathbf{a})$ - The gradient vector
$\mathrm{H} = \nabla\nabla f(\mathbf{a})$ - The hessian matrix
1_{ij}: Single-entry matrix; 1 at (i,j) and zero elsewhere
$\frac{\partial(\mathbf{x}^{\mathrm{T}}\mathrm{G})}{\partial \mathrm{G}_i} = \mathbf{x}^{\mathrm{T}} \frac{\partial(\mathrm{G})}{\partial \mathrm{G}_i} = \mathbf{x}^{\mathrm{T}} 1_i$
$\frac{\partial(\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x})}{\partial \mathrm{H}_{ij}} = \mathbf{x}^{\mathrm{T}} \frac{\partial(\mathrm{H})}{\partial \mathrm{H}_{ij}} \mathbf{x} = \mathbf{x}^{\mathrm{T}} 1_{ij} \mathbf{x}$

$$E = \sum_{i=-n}^{n} \sum_{j=-n}^{n} \sum_{k=-n}^{n} \left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x} \right)^2$$

The previous expression can be minimized to find the best values for $I_0, I_x, I_y, I_{xx}, I_{xy}$ and so on. The minimization is given below:

$$\frac{\partial E}{\partial I_0} = -2 \sum_{i=-n}^{n} \sum_{j=-n}^{n} \sum_{k=-n}^{n} \left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x} \right) = 0$$

$$\frac{\partial E}{\partial I_x} = -2 \sum_{i=-n}^{n} \sum_{j=-n}^{n} \sum_{k=-n}^{n} \mathbf{x}^{\mathrm{T}} 1_1 \left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x} \right) = 0$$

$$\frac{\partial E}{\partial I_y} = -2 \sum_{i=-n}^{n} \sum_{j=-n}^{n} \sum_{k=-n}^{n} \mathbf{x}^{\mathrm{T}} 1_2 \left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x} \right) = 0$$

$$\frac{\partial E}{\partial I_z} = -2 \sum_{i=-n}^{n} \sum_{j=-n}^{n} \sum_{k=-n}^{n} \mathbf{x}^{\mathrm{T}} 1_3 \left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x} \right) = 0$$

$$\frac{\partial E}{\partial I_{xx}} = -\frac{2}{2!} \sum_{i=-n}^{n} \sum_{j=-n}^{n} \sum_{k=-n}^{n} \left(\mathbf{x}^{\mathrm{T}} 1_{(1,1)} \mathbf{x}\right) \left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x} \right) = 0$$

$$\frac{\partial E}{\partial I_{yy}} = - \sum_{i=-n}^{n} \sum_{j=-n}^{n} \sum_{k=-n}^{n} \left(\mathbf{x}^{\mathrm{T}} 1_{(2,2)} \mathbf{x}\right) \left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x} \right) = 0$$

$$\frac{\partial E}{\partial I_{zz}} = - \sum_{i=-n}^{n} \sum_{j=-n}^{n} \sum_{k=-n}^{n} \left(\mathbf{x}^{\mathrm{T}} 1_{(3,3)} \mathbf{x}\right) \left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x} \right) = 0$$

$$\frac{\partial E}{\partial I_{xy}} = - \sum_{i=-n}^{n} \sum_{j=-n}^{n} \sum_{k=-n}^{n} \left(\mathbf{x}^{\mathrm{T}} 1_{(1,2),(2,1)} \mathbf{x}\right) \left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x} \right) = 0$$

$$\frac{\partial E}{\partial I_{xz}} = -\sum_{i=-n}^{n}\sum_{j=-n}^{n}\sum_{k=-n}^{n}\left(\mathbf{x}^{\mathrm{T}}1_{(1,3),(3,1)}\mathbf{x}\right)\left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x}\right) = 0$$

$$\frac{\partial E}{\partial I_{yz}} = -\sum_{i=-n}^{n}\sum_{j=-n}^{n}\sum_{k=-n}^{n}\left(\mathbf{x}^{\mathrm{T}}1_{(2,3),(3,2)}\mathbf{x}\right)\left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x}\right) = 0$$

Expanding the previous equations we obtain:

$$\frac{\partial E}{\partial I_0} = -2\sum_{i=-n}^{n}\sum_{j=-n}^{n}\sum_{k=-n}^{n}\left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x}\right) = 0$$

$$\sum_{i,j,k}\left(I(i,j,k) - I_0 - (iI_x + jI_y + kI_z) - \frac{1}{2!}(i^2I_{xx} + j^2I_{yy} + k^2I_{zz} + 2ijI_{xy} + 2ikI_{xz} + 2jkI_{yz})\right)$$

$$\begin{gathered}\textstyle\sum_{i,j,k} I(i,j,k) - I_0\sum_{i,j,k} - (I_x\sum_i i\sum_{j,k} + I_y\sum_j j\sum_{ik} + I_z\sum_k k\sum_{ij}) \\ \textstyle -\frac{1}{2!}\left(I_{xx}\sum_i i^2\sum_{ij} + I_{yy}\sum_j j^2\sum_{ik} + I_{zz}\sum_k k^2\sum_{ij} + 2I_{xy}\sum_{ij} ij\sum_k + 2I_{xz}\sum_{ik} ik\sum_j + 2I_{yz}\sum_{jk} jk\sum_i\right)\end{gathered} \tag{4}$$

$$\frac{\partial E}{\partial I_x} = -2\sum_{i=-n}^{n}\sum_{j=-n}^{n}\sum_{k=-n}^{n}\mathbf{x}^{\mathrm{T}}1_1\left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x}\right) = 0$$

$$\sum_{i,j,k} i\left(I(i,j,k) - I_0 - (iI_x + jI_y + kI_z) - \frac{1}{2!}(i^2I_{xx} + j^2I_{yy} + k^2I_{zz} + 2ijI_{xy} + 2ikI_{xz} + 2jkI_{yz})\right)$$

$$\begin{gathered}\textstyle\sum_i i\sum_{jk} I(i,j,k) - I_0\sum_i i\sum_{j,k} - (I_x\sum_i i^2\sum_{j,k} + I_y\sum_j j\sum_i i\sum_k + I_z\sum_k k\sum_i i\sum_j) \\ \textstyle -\frac{1}{2!}[I_{xx}\sum_i i^3\sum_{jk} + I_{yy}\sum_j j^2\sum_i i\sum_k + I_{zz}\sum_k k^2\sum_i i\sum j \\ \textstyle +2I_{xy}\sum_{ij} i^2j\sum_k + 2I_{xz}\sum_{ik} i^2k\sum_j + 2I_{yz}\sum_{jk} jk\sum_i i]\end{gathered} \tag{5}$$

$$\frac{\partial E}{\partial I_{xx}} = -\frac{2}{2!}\sum_{i=-n}^{n}\sum_{j=-n}^{n}\sum_{k=-n}^{n}\left(\mathbf{x}^{\mathrm{T}}1_{(1,1)}\mathbf{x}\right)\left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x}\right) = 0$$

$$\sum_{i,j,k} i^2\left(I(i,j,k) - I_0 - (iI_x + jI_y + kI_z) - \frac{1}{2!}(i^2I_{xx} + j^2I_{yy} + k^2I_{zz} + 2ijI_{xy} + 2ikI_{xz} + 2jkI_{yz})\right)$$

$$\begin{gathered}\textstyle\sum_i i^2\sum_{jk} I(i,j,k) - I_0\sum_i i^2\sum_{j,k} - (I_x\sum_i i^3\sum_{j,k} + I_y\sum_j j\sum_i i^2\sum_k + I_z\sum_k k\sum_i i^2\sum_j) \\ \textstyle -\frac{1}{2!}[I_{xx}\sum_i i^4\sum_{jk} + I_{yy}\sum_j j^2\sum_i i^2\sum_k + I_{zz}\sum_k k^2\sum_i i^2\sum j \\ \textstyle +2I_{xy}\sum_{ij} i^3j\sum_k + 2I_{xz}\sum_{ik} i^3k\sum_j + 2I_{yz}\sum_{jk} jk\sum_i i^2]\end{gathered} \tag{6}$$

$$\frac{\partial E}{\partial I_{xy}} = -\sum_{i=-n}^{n}\sum_{j=-n}^{n}\sum_{k=-n}^{n}\left(\mathbf{x}^{\mathrm{T}}1_{(1,2),(2,1)}\mathbf{x}\right)\left(I(\mathbf{x}) - I_0 - \mathbf{x}^{\mathrm{T}}\mathrm{G} - \frac{1}{2!}\mathbf{x}^{\mathrm{T}}\mathrm{H}\mathbf{x}\right) = 0$$

$$\sum_{i,j,k} 2ij \left(I(i,j,k) - I_0 - (iI_x + jI_y + kI_z) - \frac{1}{2!}(i^2 I_{xx} + j^2 I_{yy} + k^2 I_{zz} + 2ijI_{xy} + 2ikI_{xz} + 2jkI_{yz}) \right)$$

$$\begin{gathered}2\sum_i i \sum_j j \sum_k I(i,j,k) - 2I_0 \sum_i i \sum_j j \sum k - 2(I_x \sum_i i^2 \sum_j j \sum_k + I_y \sum_j j^2 \sum_i i \sum_k + I_z \sum_k k \sum_i i \sum_j j) \\ -(I_{xx} \sum_i i^3 \sum_j j \sum_k + I_{yy} \sum_j j^3 \sum_i i \sum_k + I_{zz} \sum_k k^2 \sum_i i \sum_j j \\ +2I_{xy} \sum_i i^2 \sum_j j^2 \sum_k + 2I_{xz} \sum_i i^2 \sum_k k \sum_j j + 2I_{yz} \sum_j j^2 \sum_k k \sum_i i)\end{gathered} \tag{7}$$

The previous equation indicates that for a second order approximation our Taylor-basis is 9-dimensional
The system of linear equations can be solved using the following matrix:

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} I_0 & I_x & I_y & I_z & I_{xx} & I_{yy} & I_{zz} & I_{xy} & I_{xz} & I_{yz} \end{array} \\
\begin{array}{c}
\frac{\delta E}{\delta I_0} \\ \frac{\delta E}{\delta I_x} \\ \frac{\delta E}{\delta I_y} \\ \frac{\delta E}{\delta I_z} \\ \frac{\delta E}{\delta I_{xx}} \\ \frac{\delta E}{\delta I_{yy}} \\ \frac{\delta E}{\delta I_{zz}} \\ \frac{\delta E}{\delta I_{xy}} \\ \frac{\delta E}{\delta I_{xz}} \\ \frac{\delta E}{\delta I_{yz}}
\end{array}
&
\begin{pmatrix}
\sum_{ijk} & \sum_i i \sum_{jk} & \sum_j j \sum_{ik} & \sum_k k \sum_{ij} & \frac{1}{2}\sum_i i^2 \sum_{jk} & \frac{1}{2}\sum_j j^2 \sum_{ik} & \frac{1}{2}\sum_k k^2 \sum_{ij} & \sum_i i \sum_j j \sum_k & \sum_i i \sum_k k \sum_j & \sum_j j \sum_k k \sum_i \\
\sum_i i \sum_{jk} & \sum_i i^2 \sum_{jk} & \sum_i i \sum_j j \sum_k & \sum_i i \sum_k k \sum_j & \frac{1}{2}\sum_i i^3 \sum_{jk} & \frac{1}{2}\sum_i i \sum_j j^2 \sum_k & \frac{1}{2}\sum_i i \sum_k k^2 \sum_j & \sum_i i^2 \sum_j j \sum_k & \sum_i i^2 \sum_k k \sum_j & \sum_i i \sum_j j \sum_k k \\
\sum_j j \sum_{ik} & \sum_i i \sum_j j \sum_k & \sum_j j^2 \sum_{ik} & \sum_j j \sum_k k \sum_i & \frac{1}{2}\sum_i i^2 \sum_j j \sum_k & \frac{1}{2}\sum_j j^3 \sum_{ik} & \frac{1}{2}\sum_j j \sum_k k^2 \sum_i & \sum_i i \sum_j j^2 \sum_k & \sum_i i \sum_j j \sum_k k & \sum_j j^2 \sum_k k \sum_i \\
\sum_k k \sum_{ij} & \sum_i i \sum_k k \sum_j & \sum_j j \sum_k k \sum_i & \sum_k k^2 \sum_{ij} & \frac{1}{2}\sum_i i^2 \sum_k k \sum_j & \frac{1}{2}\sum_j j^2 \sum_k k \sum_i & \frac{1}{2}\sum_k k^3 \sum_{ij} & \sum_i i \sum_j j \sum_k k & \sum_i i \sum_k k^2 \sum_j & \sum_j j \sum_k k^2 \sum_i \\
\sum_i i^2 \sum_{jk} & \sum_i i^3 \sum_{jk} & \sum_i i^2 \sum_j j \sum_k & \sum_i i^2 \sum_k k \sum_j & \frac{1}{2}\sum_i i^4 \sum_{jk} & \frac{1}{2}\sum_i i^2 \sum_j j^2 \sum_k & \frac{1}{2}\sum_i i^2 \sum_k k^2 \sum_j & \sum_i i^3 \sum_j j \sum_k & \sum_i i^3 \sum_k k \sum_j & \sum_i i^2 \sum_j j \sum_k k \\
\sum_j j^2 \sum_{ik} & \sum_i i \sum_j j^2 \sum_k & \sum_j j^3 \sum_{ik} & \sum_j j^2 \sum_k k \sum_i & \frac{1}{2}\sum_i i^2 \sum_j j^2 \sum_k & \frac{1}{2}\sum_j j^4 \sum_{ik} & \frac{1}{2}\sum_k k^2 \sum_j j^2 \sum_i & \sum_i i \sum_j j^3 \sum_k & \sum_i i \sum_j j^2 \sum_k k & \sum_j j^3 \sum_k k \sum_i \\
\sum_k k^2 \sum ij & \sum_i i \sum_k k^2 \sum_j & \sum_j j \sum_k k^2 \sum_i & \sum_k k^3 \sum_{ij} & \frac{1}{2}\sum_i i^2 \sum_k k^2 \sum_j & \frac{1}{2}\sum_j j^2 \sum_k k^2 \sum_i & \frac{1}{2}\sum_k k^4 \sum_{ij} & \sum_i i \sum_j j \sum_k k^2 & \sum_i i \sum_k k^3 \sum_j & \sum_j j \sum_k k^3 \sum_i \\
2\sum_i i \sum_j j \sum_k & 2\sum_i i^2 \sum_j j \sum_k & 2\sum_i i \sum_j j^2 \sum_k & 2\sum_i i \sum_j j \sum_k k & \sum_i i^3 \sum_j j \sum_k & \sum_i i \sum_j j^3 \sum_k & \sum_i i \sum_j j \sum_k k^2 & 2\sum_i i^2 \sum_j j^2 \sum_k & 2\sum_i i^2 \sum_j j \sum_k k & 2\sum_i i \sum_j j^2 \sum_k k \\
2\sum_i i \sum_k k \sum_j & 2\sum_i i^2 \sum_k k \sum_j & 2\sum_i i \sum_j j \sum_k k & 2\sum_i i \sum_k k^2 \sum_j & \sum_i i^3 \sum_k k \sum_j & \sum_i i \sum_j j^2 \sum_k k & \sum_i i \sum_k k^3 \sum_j & 2\sum_i i^2 \sum_j j \sum_k k & 2\sum_i i^2 \sum_k k^2 \sum_j & 2\sum_i i \sum_j j \sum_k k^2 \\
2\sum_j j \sum_k k \sum_i & 2\sum_i i \sum_j j \sum_k k & 2\sum_j j^2 \sum_k k \sum_i & 2\sum_j j \sum_k k^2 \sum_i & \sum_i i^2 \sum_j j \sum_k k & \sum_j j^3 \sum_k k \sum_i & \sum_j j \sum_k k^3 \sum_i & 2\sum_i i \sum_j j^2 \sum_k k & 2\sum_i i \sum_j j \sum_k k^2 & 2\sum_j j^2 \sum_k k^2 \sum_i
\end{pmatrix}
\end{array}
=
\begin{pmatrix}
\sum_{ijk} I(i,j,k) \\
\sum_i i \sum_{jk} I(i,j,k) \\
\sum_j j \sum_{ik} I(i,j,k) \\
\sum_k k \sum_{ij} I(i,j,k) \\
\sum_i i^2 \sum_{jk} I(i,j,k) \\
\sum_j j^2 \sum ikI(i,j,k) \\
\sum_k k^2 \sum_{ij} I(i,j,k) \\
2\sum_i i \sum_j j \sum_k I(i,j,k) \\
2\sum_i i \sum_k k \sum_j I(i,j,k) \\
2\sum_j j \sum_k k \sum_i I(i,j,k)
\end{pmatrix}
$$

Misssing! What are the kernels? Well we could run derivatives of gaussians that are already implemented - not the xy. Also, should we go above 2nd derivative - so we get a larger basis to compare Harr Wavelets? or other wavelets?